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Gaussian Limiting Behavior of the Rescaled Solution to the Linear Korteweg–de Vries Equation with Random Initial Conditions

Luisa Beghin,[1] **Viktoria P. Knopova,**[2] **Nikolai N. Leonenko,**[2] **and Enzo Orsingher**[1]

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We analyze the asymptotic behavior of the rescaled solution to the linear Korteweg–de Vries equation when the initial conditions are supposed to be random and weakly dependent. By means of the method of moments we prove the Gaussianity of the limiting process and we present its correlation function. The same technique is applied to the analysis of another third-order heat-type equation.

KEY WORDS: Linear Korteweg–de Vries equation; third-order heat-type equation; random initial conditions; scaling limits; weak dependence; Hermite expansion; Airy function.

1. INTRODUCTION

In this paper we consider the asymptotic behavior of suitably rescaled solutions of third-order heat-type equations with random initial conditions, represented by transformed, stationary, Gaussian processes.

Third-order heat-type equations emerge in the context of trimolecular chemical reactions (Gardiner (1985), p. 299) and also as linear approximations of the celebrated Korteweg–de Vries (hereafter KdV) equation (see, for example, Drazin and Johnson (1989), p. 18).

[1] Dipartimento di Statistica, Probabilità e Statistiche Applicate, Università degli Studi di Roma "La Sapienza," 00185 Rome, Italy.
[2] Department of Mathematics, Kiev University (National), 252601 Kiev, Ukraine.

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Equations of the form

$$\frac{\partial u}{\partial t} = c_3 \frac{\partial^3 u}{\partial x^3}, \qquad x \in \mathbb{R}, \quad t > 0 \tag{1.1}$$

where $c_3 = \pm 1$, have been dealt with in Orsingher (1991) in connection with the study of precesses governed by signed measures. The case where $c_3 = \pm i$ has been analyzed in Hochberg and Orsingher (1994).

Note that general higher-order heat-type equations have been considered by many authors: Krylov (1960), Daletski *et al.* (1961), Myiamoto (1966), Hochberg (1978), Funaki (1979), Hochberg and Orsingher (1996), Orsingher and Zhao (1999) and the references therein.

The rescaling procedures for partial differential equations with random data have been studied by Avellaneda and Majda (1990), Ratanov *et al.* (1991), Bulinski and Molchanov (1991), Deriev and Leonenko (1997), Leonenko and Woyczynski (1998a, b), Gaudron (1998), Anh and Leonenko (1999), Leonenko (1999) and many others. Our paper considers the solution of the linear KdV equation with weakly dependent, random initial conditions. Thus it is in the mainstream of the series of papers devoted to the Burgers equation (Albeverio *et al.* (1994), Deriev and Leonenko (1997), Leonenko and Woyczynski (1998a)) and to the classical heat equation (Leonenko and Woyczynski (1998b)), where weakly dependent, random initial conditions are assumed.

Our results here are limit theorems for non-linear transformations of Gaussian random fields with weak dependence similar to those presented in Breuer and Major (1983), Ivanov and Leonenko (1989), pp. 70–77, and Leonenko (1999), pp. 227–243.

In Section 2 we present the structure of the correlation function of the solution, with general random initial conditions, in terms of Airy functions.

By assuming suitable restrictions on the process representing the initial conditions, we study, in Section 3 the asymptotic behaviour of the rescaled solutions.

In principle, our technique can be applied also to the analysis of the general KdV equation with random initial conditions.

The case of the randomly forced KdV equation, that is

$$\frac{\partial u}{\partial t} + \sigma u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = w(t) \tag{1.2}$$

where $w(t)$, $t \in R$ is a stochastic process, has been studied by Orlowski and Sobszyk (1989).

2. SECOND ORDER ANALYSIS OF THE LINEAR KDV EQUATION WITH RANDOM INITIAL DATA

We first consider the linear KdV equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3}, \qquad t > 0, \quad x \in \mathbb{R} \tag{2.1}$$

subject to the random initial condition

$$u(0, x) = \eta(x) \tag{2.2}$$

Clearly Eq. (2.1) coincides with (1.2) where $\sigma = 0$, $w(t) = 0$.

The random process $\eta = \eta(x, \omega)$, $\omega \in \Omega$, $x \in \mathbb{R}$, defined on a suitable, complete probability space $(\Omega, \mathcal{F}, P)$, is assumed to be measurable, mean-square continuous, with mean $E\eta(x) = M(x)$.

The process $\eta(x) - M(x)$ is weakly stationary and then $\eta(x)$ possesses the following spectral representation (P-a.s.)

$$\eta(x) = M(x) + \int_{-\infty}^{+\infty} \exp\{i\lambda x\}\, Z(d\lambda) \tag{2.3}$$

where $Z = Z(\Delta)$, $\Delta \in \mathcal{B}(\mathbb{R})$ is a complex-valued random measure such that $E\,|Z(\Delta)|^2 = F(\Delta)$, F is the spectral (bounded) measure on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and the stochastic integral in (2.3) is viewed as an L_2 integral with control measure F.

The covariance function of the process η, by the Bochner–Khinchin Theorem, possesses the spectral representation

$$\begin{aligned} B(x) &= cov(\eta(y), \eta(y + x)) \\ &= \int_{-\infty}^{+\infty} \exp\{i\lambda x\}\, F(d\lambda), \qquad x \in \mathbb{R} \end{aligned} \tag{2.4}$$

The fundamental solution to Eq. (2.1) can be represented in one of the following forms (see Drazin and Johnson (1989), Orsingher (1991)):

$$\begin{aligned} u_0(t, x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\alpha x - i\alpha^3 t}\, d\alpha = \frac{1}{\pi} \int_0^{\infty} \cos(\alpha x + \alpha^3 t)\, d\alpha \\ &= \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt[3]{3t}} Ai\left(\frac{x}{\sqrt[3]{3t}}\right) \end{aligned} \tag{2.5}$$

where

$$Ai(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos\left(\alpha x + \frac{\alpha^3}{3}\right) d\alpha, \qquad x \in \mathbb{R}$$

is the Airy function of first kind (see Bleistein and Handelsman (1986)).

By means of the steepest-descent method it has been proved (see Accetta and Orsingher (1997)) that the fundamental solution has the following asymptotic behaviour

$$u_0(t, x) \sim \frac{x^{-1/4} t^{-1/4}}{2\sqrt{\pi}\sqrt[4]{3}} \exp\left\{ -\frac{2}{3\sqrt{3}} x^{3/2} t^{-1/2} \right\}, \qquad x \to +\infty$$

and

$$u_0(t, x) \sim \frac{|x|^{-1/4}\, t^{-1/4}}{\sqrt{\pi}\sqrt[4]{3}} \cos\left\{ \frac{2}{3\sqrt{3}} |x|^{3/2}\, t^{-1/2} - \frac{\pi}{4} \right\}, \qquad x \to -\infty$$

Thus, for any $t > 0$, $u_0(t, x)$ converges to zero exponentially fast as $x \to +\infty$ and oscillating as $x \to -\infty$, alternating negative and positive values. Thus $u_0(t, x)$ is asymmetric and signed. We note that the fundamental solution to the second-order equation is non-negative and to the fourth-order one is signed, nevertheless both of them are symmetric.

By the linearity of Eq. (2.1) the solution to the initial-value problem (2.1)–(2.2) can be written down as follows

$$\begin{aligned}
u(t, x) &= \int_{-\infty}^{+\infty} \eta(x - y)\, u_0(t, y)\, dy \\
&= \int_{-\infty}^{+\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \eta(x - y) \exp\{-i\alpha y - i\alpha^3 t\}\, d\alpha\, dy \\
&= (\text{by } (2.3)) \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} M(x - y) \exp\{-i\alpha y - i\alpha^3 t\}\, d\alpha\, dy \\
&\quad + \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\alpha \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} Z(d\lambda) \exp\{i\lambda x - i\lambda y - i\alpha y - i\alpha^3 t\}
\end{aligned}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{M(x-y)}{\sqrt[3]{3t}} A_i\left(\frac{y}{\sqrt[3]{3t}}\right) dy$$

$$+ \int_{-\infty}^{+\infty} d\alpha \int_{-\infty}^{+\infty} Z(d\lambda) \exp\{i\lambda x - i\alpha^3 t\}\ \delta(\alpha+\lambda)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{M(x-y)}{\sqrt[3]{3t}} Ai\left(\frac{y}{\sqrt[3]{3t}}\right) dy + \int_{-\infty}^{+\infty} \exp\{i\lambda x + i\lambda^3 t\}\ dZ(\lambda) \tag{2.6}$$

where

$$\delta(\alpha+\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix(\alpha+\lambda)}\, dx$$

is the Dirac's delta function.

It is straightforward from (2.6) that

$$Eu(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{M(x-y)}{\sqrt[3]{3t}} A_i\left(\frac{y}{\sqrt[3]{3t}}\right) dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{M(y)}{\sqrt[3]{3t}} Ai\left(\frac{x-y}{\sqrt[3]{3t}}\right) dy$$

We assume that the random process η has sample paths such that the function (2.8) satisfies (2.1) with probability one and $Eu(t, x)$ exists for any t and x.

From (2.6) it is possible to extract the expression of the covariance function of the random field $u = u(t, x)$:

$$cov(u(t,x), u(t',x')) = \int_{-\infty}^{+\infty} \exp\{i\lambda(x-x') + i\lambda^3(t-t')\}\ dF(\lambda)$$

$$= \int_{-\infty}^{+\infty} \cos[\lambda(x-x') + \lambda^3(t-t')]\ dF(\lambda) \tag{2.7}$$

Formula (2.7) shows that the random field u is stationary with respect to space and time.

The covariance function (2.7) can be written down also in terms of the Airy function and of the covariance function of the process η. By using the real representation of the solution, namely

$$u(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \eta(y) \frac{1}{\sqrt[3]{3t}} Ai\left(\frac{x-y}{\sqrt[3]{3t}}\right) dy \tag{2.8}$$

for $t > t'$ we have

$$
\begin{aligned}
&cov(u(t,x), u(t',x')) \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} cov(\eta(y),\eta(y'))\frac{1}{\pi}\, Ai\left(\frac{x-y}{\sqrt[3]{3t}}\right) Ai\left(\frac{x'-y'}{\sqrt[3]{3t'}}\right)\frac{dy}{\sqrt[3]{3t}}\frac{dy'}{\sqrt[3]{3t'}} \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} B(z)\, Ai\left(\frac{u}{\sqrt[3]{3t}}\right) Ai\left(\frac{x'-x+z+u}{\sqrt[3]{3t'}}\right)\frac{dz}{\sqrt[3]{3t}}\frac{du}{\sqrt[3]{3t'}} \\
&= \int_{-\infty}^{+\infty} B(z)\,dz \int_{-\infty}^{+\infty} du \left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\{-i\alpha u - i\alpha^3 t\}\,d\alpha\right\} \\
&\times\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\{-i\alpha'(x'-x+z+u)-i\alpha'^3 t'\}\,d\alpha'\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} B(z)\,dz\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\{-i\alpha^3 t - i\alpha'(x'-x+z)-i\alpha'^3 t'\} \\
&\times\delta(\alpha+\alpha')\,d\alpha\,d\alpha' \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} B(z)\,dz\int_{-\infty}^{+\infty}\exp\{-i\alpha^3 t + i\alpha^3 t' + i\alpha(x'-x+z)\}\,d\alpha \\
&= \int_{-\infty}^{+\infty} B(z)\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt[3]{3(t-t')}}\, Ai\left(\frac{x-x'-z}{\sqrt[3]{3(t-t')}}\right) dz
\end{aligned}
\tag{2.9}
$$

Similarly, for $t < t'$, we have

$$
\begin{aligned}
&cov(u(t,x), u(t',x')) \\
&= \int_{-\infty}^{+\infty} B(z)\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt[3]{3(t'-t)}}\, Ai\left(\frac{x'-x+z}{\sqrt[3]{3(t'-t)}}\right) dz
\end{aligned}
\tag{2.10}
$$

Finally, for $t = t'$, we have

$$cov(u(t,x)\, u(t',x')) = B(|x-x'|) \tag{2.11}$$

Little changes are necessary for the analysis of the random field emerging as solution to the equation

$$\frac{\partial v}{\partial t} = \frac{\partial^3 v}{\partial x^3}, \qquad t>0, \quad x\in\mathbb{R} \tag{2.12}$$

subject to the random initial condition

$$v(0, x) = \eta(x) \tag{2.13}$$

The fundamental solution in this case reads

$$\begin{aligned} v_0(t, x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\alpha x + i\alpha^3 t} \, d\alpha = \frac{1}{\pi} \int_0^{\infty} \cos(\alpha x - \alpha^3 t) \, d\alpha \\ &= \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt[3]{3t}} Ai\left(-\frac{x}{\sqrt[3]{3t}} \right), \qquad t > 0, \quad x \in \mathbb{R} \end{aligned} \tag{2.14}$$

In force of the linearity of Eq. (2.12), the general solution to the initial-value problem (2.12)–(2.13) can be written down as follows:

$$\begin{aligned} v(t, x) &= \int_{-\infty}^{+\infty} \eta(x - y) \, v_0(t, y) \, dy \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{M(x - y)}{\sqrt[3]{3t}} Ai\left(-\frac{y}{\sqrt[3]{3t}} \right) dy + \int_{-\infty}^{+\infty} \exp\{ i\lambda x - i\lambda^3 t \} \, dZ(\lambda) \end{aligned} \tag{2.15}$$

We assume that the random process η has sample paths such that the function (2.15) satisfies (2.12) with probability one and $Eu(t, x)$ exists for any t and x.

Paralleling the calculations leading to (2.9), (2.10) and (2.11) we obtain

$$cov(v(t, x), v(t', x'))$$

$$= \begin{cases} B(|x - x'|), & \text{for} \quad t = t' \\ \displaystyle\int_{-\infty}^{+\infty} B(z) \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt[3]{3(t - t')}} Ai\left(\frac{x' - x + z}{\sqrt[3]{3(t - t')}} \right) dz, & \text{for} \quad t > t' \\ \displaystyle\int_{-\infty}^{+\infty} B(z) \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt[3]{3(t' - t)}} Ai\left(\frac{x - x' - z}{\sqrt[3]{3(t' - t)}} \right) dz, & \text{for} \quad t < t' \end{cases} \tag{2.16}$$

3. THE ASYMPTOTIC BEHAVIOUR OF THE RESCALED SOLUTIONS

We now consider the rescaled solution to the linear KdV equation (2.1) in the case where the initial condition is represented by the following process

$$\eta(x) = M(x) + G(\xi(x)), \qquad x \in \mathbb{R} \tag{3.1}$$

where M and G are real-valued, deterministic functions and $\xi = \xi(x)$ is a stationary, Gaussian process.

Our analysis is performed under the following assumptions:

A. The random process $\xi = \xi(x)$ is real, measurable, mean-square continuous, stationary and Gaussian with $E\xi(x) = 0$, $E\xi^2(x) = 1$ and covariance function $R(x)$.

B. The function G: $\mathbb{R} \to \mathbb{R}$ is non-linear and such that $EG^2(\xi(0)) < \infty$.

As it is well-known, the function G can be expanded in series as follows

$$G(u) = \sum_{k=0}^{\infty} \frac{C_k H_k(u)}{k!}, \qquad C_k = \int_{-\infty}^{+\infty} G(u)\, \varphi(u)\, H_k(u)\, du \tag{3.2}$$

where

$$H_m(u) = (-1)^m\, e^{u^2/2} \frac{d^m}{du^m} e^{-u^2/2}, \qquad u \in \mathbb{R}, \quad m = 0, 1, 2, \ldots$$

are the Chebyshev–Hermite polynomials, which form a complete, orthogonal basis in the Hilbert space $L_2(\mathbb{R}, \varphi(u)\, du)$ with

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{u^2}{2} \right\}, \qquad u \in \mathbb{R}$$

We now further assume for the function G the following condition

C. G is such that there exist an integer value m for which $C_m \neq 0$ and $C_1 = C_2 = \cdots = C_{m-1} = 0$.

We pass on to the statement of our main result

Theorem 3.1. Suppose, that the random process η has sample paths such that the function (2.6) satisfies (2.1) with probability one and $Eu(t, x)$ exists for any t and x.

Let $u = u(t, x)$, $t > 0$, $x \in \mathbb{R}$, be the solution to the initial-value problem (2.1)–(2.2) with the random initial data (3.1) satisfying the conditions A–C, together with

$$\int_{-\infty}^{+\infty} |R(z)|^m\, dz < \infty \tag{3.3}$$

then the finite-dimensional distributions of the random fields

$$U_\varepsilon(t,x)=\varepsilon^{-1/6}\left[u\left(\frac{t}{\varepsilon},\frac{x}{\sqrt[3]{\varepsilon}}\right)-Eu\left(\frac{t}{\varepsilon},\frac{x}{\sqrt[3]{\varepsilon}}\right)\right],\qquad t>0,\quad x\in\mathbb{R},\quad \varepsilon>0 \tag{3.4}$$

converge weakly, as $\varepsilon\to 0$, to the finite-dimensional distributions of the Gaussian random field $U(t,x)$, $t>0$, $x\in\mathbb{R}$, (stationary in space and time) with mean $EU(t,x)=0$ and covariance function

$$EU(t,x)\,U(t',x')=\begin{cases}\dfrac{\sigma^2}{\sqrt{\pi}}\dfrac{1}{\sqrt[3]{3(t-t')}}\,Ai\left(\dfrac{x-x'}{\sqrt[3]{3(t-t')}}\right), & \text{for } t>t'\\[2ex] \dfrac{\sigma^2}{\sqrt{\pi}}\dfrac{1}{\sqrt[3]{3(t'-t)}}\,Ai\left(\dfrac{x'-x}{\sqrt[3]{3(t'-t)}}\right), & \text{for } t<t'\\[2ex] \sum_{k=m}^{\infty}\dfrac{C_k^2}{k!}\,R^k(|x-x'|), & \text{for } t=t'\end{cases} \tag{3.5}$$

where

$$\sigma^2=\sum_{k=m}^{\infty}\frac{C_k^2}{k!}\int_{-\infty}^{+\infty}R^k(z)\,dz \tag{3.6}$$

Hint of the Proof. For the random variables

$$\begin{aligned}\zeta_\varepsilon&=\sum_{j=1}^{k}\lambda_j U_\varepsilon(t_j,x_j)\\ &=\sum_{j=1}^{k}\lambda_j\int_{-\infty}^{+\infty}\eta(y)\,u_0\left(\frac{t_j}{\varepsilon},\frac{x_j}{\sqrt[3]{\varepsilon}}-y\right)\frac{1}{\sqrt[6]{\varepsilon}}\,dy\\ &=\sum_{j=1}^{k}\lambda_j\int_{-\infty}^{+\infty}\sum_{r=m}^{\infty}\frac{C_r}{r!}H_r(\xi(y))\,u_0\left(\frac{t_j}{\varepsilon},\frac{x_j}{\sqrt[3]{\varepsilon}}-y\right)\frac{1}{\sqrt[6]{\varepsilon}}\,dy\\ &=\sum_{j=1}^{k}\lambda_j\left[\sum_{r=m}^{N}+\sum_{r=N+1}^{\infty}\right]\frac{C_r}{r!}\int_{-\infty}^{+\infty}H_r(\xi(y))\\ &\quad\times\frac{1}{\sqrt{\pi}}\frac{\sqrt[6]{\varepsilon}}{\sqrt[3]{3t_j}}\,Ai\left(\frac{x_j-y\sqrt[3]{\varepsilon}}{\sqrt[3]{3t_j}}\right)dy\\ &=\zeta'_\varepsilon[r\leqslant N]+\zeta''_\varepsilon[r>N]=\zeta'_\varepsilon+\zeta''_\varepsilon\end{aligned}$$

the convergence to

$$\zeta = \sum_{j=1}^{k} \lambda_j U(t_j, x_j)$$

must be studied.

To prove that ζ''_ε converges to zero, we need only some estimation of $var(\zeta''_\varepsilon)$.

In the analysis of the convergence of ζ'_ε the so called *method of the diagram* must be applied; see Taqqu (1977), Ivanov and Leonenko (1989), pp. 70–76, Breuer and Major (1983), Leonenko (1999), pp. 225–243, for general references, and Deriev and Leonenko (1997), for a specific application of the method.

Remark 3.1. In view of the asymptotic behaviour of the Airy function, we can conclude that the limiting covariance function (3.5) decreases exponentially fast as $x - x' \to +\infty$ and decreases oscillating as $x - x' \to -\infty$ (when $t > t'$). We can interpret this behaviour as a short-range dependence for $x - x' \to +\infty$ and a long-range dependence for $x - x' \to -\infty$.

For $t < t'$, the same behaviour of the covariance function is recorded for $x' - x \to \pm\infty$.

For the rescaled random field related to Eq. (2.12) we have the following result, which can be proved similarly to the previous one:

Theorem 3.2. Let $v = v(t, x)$, $t > 0$, $x \in \mathbb{R}$, be the solution to the initial-value problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^3 v}{\partial x^3} \\ v(0, x) = \eta(x) \end{cases} \tag{3.7}$$

where the random initial data η satisfies the conditions A–C and the assumption (3.3), and has sample paths such that function (2.12) satisfies (3.7) with probability one and $Ev(t, x)$ exists for any t and x. Then the finite-dimensional distributions of the random fields

$$V_\varepsilon(t, x) = \varepsilon^{-1/6}\left[v\left(\frac{t}{\varepsilon}, \frac{x}{\sqrt[3]{\varepsilon}}\right) - Ev\left(\frac{t}{\varepsilon}, \frac{x}{\sqrt[3]{\varepsilon}}\right)\right], \qquad t > 0, \quad x \in \mathbb{R}, \quad \varepsilon > 0 \tag{3.8}$$

converge weakly, as $\varepsilon \to 0$, to the finite-dimensional distributions of the Gaussian random field (stationary in space and time) with mean $EV(t, x) = 0$ and covariance function

$$EV(t, x)\, V(t', x') = \begin{cases} \dfrac{\sigma^2}{\sqrt{\pi}} \dfrac{1}{\sqrt[3]{3(t-t')}} Ai\left(\dfrac{x'-x}{\sqrt[3]{3(t-t')}}\right), & \text{for} \quad t > t' \\ \dfrac{\sigma^2}{\sqrt{\pi}} \dfrac{1}{\sqrt[3]{3(t'-t)}} Ai\left(\dfrac{x-x'}{\sqrt[3]{3(t'-t)}}\right), & \text{for} \quad t < t' \\ \sum_{k=m}^{\infty} \dfrac{C_k^2}{k!} R^k(|x-x'|), & \text{for} \quad t = t' \end{cases} \tag{3.9}$$

where σ^2 is defined by (3.6).

Remark 3.2. The above theorems should be compared with the result of Bulinski and Molchanov (1991), Albeverio *et al.* (1994), Deriev and Leonenko (1997) and Leonenko and Woyczynski (1998b). In the last one the random field $X(t, x)$, $t > 0$, $x \in \mathbb{R}$, emerging as solution to the heat equation

$$\frac{\partial h}{\partial t} = \mu \frac{\partial^2 h}{\partial x^2}, \qquad \mu > 0, \quad t > 0, \quad x \in \mathbb{R} \tag{3.10}$$

subject to the random initial condition (3.1) is examined.

Under the condition of Theorem 3.2 it is proved that the finite-dimensional distributions of the random fields

$$X_\varepsilon(t, x) = \varepsilon^{-1/4} \left[h\left(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}}\right) - Eh\left(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}}\right) \right], \qquad t > 0, \quad x \in \mathbb{R}, \quad \varepsilon > 0 \tag{3.11}$$

converge weakly to the finite-dimensional distributions of a Gaussian random field $X(t, x)$, $t > 0$, $x \in \mathbb{R}$, with mean $Ex(t, x) = 0$ and covariance function

$$Ex(t, x)\, X(t', x') = \frac{\sigma^2}{\sqrt{4\pi\mu(t+t')}} \exp\left\{ -\frac{|x-x'|^2}{4\mu(t+t')} \right\} \tag{3.12}$$

Note that the limiting field X is stationary in space but not with respect to time. This is the main difference between the limiting fields X and U.

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REFERENCES

1. G. Accetta and E. Orsingher, Asymptotic expansion of fundamental solutions of higher order heat equations, *Random Operators and Stoch. Equ.* **5**(3):217–226 (1997).
2. S. Albeverio, S. A. Molchanov, and D. Surgailis, Stratified structure of the Universe and Burgers' equation: a probabilistic approach, *Prob. Theory and Rel. Fields* **100**:457–484 (1994).
3. V. V. Anh and N. N. Leonenko, Non-Gaussian scenarios for the heat equation with singular initial conditions, *Stoch. Proc. Applic.* **84**:91–114 (1999).
4. M. Avellaneda and A. Majda, Mathematical models with exact renormalisation for turbulent transport, *Comm. Math. Phys.* **131**:381–429 (1990).
5. N. Bleistein and R. A. Handelsman, *Asymptotic Expansions of Integrals* (Dover Publ. Inc., New York, 1986).
6. P. Breuer and P. Major, Central theorem for non-linear functionals of Gaussian fields, *Journ. Multivariate Anal.* **13**:425–441 (1983).
7. A. V. Bulinski and S. A. Molchanov, Asymptotic Gaussianness of solutions to the Burgers' equation with random initial data, *Theory Prob. Appl.* **36**:217–235 (1991).
8. Yu. L. Daletsky, V. Yu. Krylov, R. A. Minlos, and V. N. Sudakov, Functional integration and measures in functional spaces, *Proc. Fourth All Union Math. Congr.*, Leningrad **2**:282–292 1961) (in Russian).
9. I. I. Deriev and N. N. Leonenko, Limit Gaussian behaviour of the solutions of the multidimensional Burgers' equation with weak-dependent initial conditions, *Acta Appl. Math.* **47**:178 (1997).
10. P. G. Drazin and R. S. Johnson, *Solitons: An Introduction* (Cambridge University Press, Cambridge, 1989).
11. T. Funaki, Probabilistic construction of the solutions of some higher-order parabolic differential equations, *Proc. Japan Acad. A* **55**:176–179 (1979).
12. C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Springer-Verlag, Berlin, 1985).
13. G. Gaudron, Scaling laws and convergence for advection-diffusion equation, *Ann. Appl. Prob.* **8**:649–663 (1998).
14. K. J. Hochberg, A signed measure on path space related to Wiener measure, *Ann. Probab.* **6**:433–458 (1978).
15. K. J. Hochberg and E. Orsingher, The arc-sine law and its analogs for processes governed by signed and complex measures, *Stoch. Proc. Appl.* **52**:273–292 (1994).
16. K. J. Hochberg and E. Orsingher, Composition of stochastic processes governed by higher-order parabolic and hyperbolic equation, *Journ. Theor. Prob.* **9**:511–532 (1996).
17. A. V. Ivanov and N. N. Leonenko, *Statistical Analysis of Random Fields* (Kluwer Academic Publishers, Dordrecht, 1989).
18. V. Yu. Krylov, Some properties of the distribution corresponding to the equation $\partial u/\partial t = (-1)^{q+1} \partial^{2p}u/\partial x^{2p}$, *Soviet Math. Dokl.* **1**:260–263 (1960).
19. N. N. Leonenko, *Limit Theorems for Random Fields with Singular Spectrum* (Kluwer Academic Publishers, Dordrecht, 1999).

20. N. N. Leonenko and W. A. Woyczynski, Exact parabolic asymptotics for singular n-D Burgers' random fields: Gaussian approximation, *Stoch. Proc. Appl.* **76**:141–165 (1998a).
21. N. N. Leonenko and W. A. Woyczynski, Scaling limits of the heat equation for singular non-Gaussian data, *Journ. Stat. Phys.* **91**(1/2):423–438 (1998b).
22. Myiamoto, An extension of certain quasi-measures, *Proc. Japan Ac. Coll.* **II-11**:70–74 (1966).
23. A. Orlowski and K. Sobczyk, Solitons and shock waves under random external noise, *Rep. Math. Phys.* **27**:59–71 (1989).
24. E. Orsingher, Processes governed by signed measures connected with third-order "heat-type" equations, *Lith. Math. J.* **31**:321–334 (1991).
25. E. Orsingher and X. Zhao, Iterated processes and their applications to higher order differential equations, *Acta Math. Sinica* **15**(2):173–180 (1999).
26. N. E. Ratanov, A. G. Shuhov, and Yu. M. Suhov, Stabilization of the statistical solutions of the parabolic equation, *Acta Appl. Math.* **22**:103–115 (1991).
27. M. S. Taqqu, Law of the iterated logarithm for sums of non-linear functions of Gaussian variables that exhibit a long-range dependence, *Z. Wahrsh. verw. Gebiete.* **40**:203–238 (1977).